An essay on the Relativistic Theory of the Non-Symmetric Field

Imposing the transposition invariance to metric-affine theories of gravity and to supergravity

Relating dark energy to the torsion of spacetime

Kurt Reichling*

version 6.9g9, January 2010

Extract

We show that imposing the transposition invariance to metric-affine theories of gravity and to supergravity leads to formally simpler and stronger field equations. Using the simplest metric-affine theory of gravity with torsion, we also show that the requirement of transposition invariance allows to physically reinterpret the ΛCDM cosmological model by relating dark energy to the torsion of spacetime.

"One should not desist from pursuing to the end the path of the relativistic field theory".
Albert Einstein, *"Ideas and Opinions"*, Bonanza Books, New York, 1954.

Throughout this paper, we refer to metric-affine theories of gravity and to supergravity. The reader is therefore supposed to be familiar with [6][8] and references therein.

The source code and other machine readable formats of the last version of this paper are available on the internet at *http://www.e-kr.org* . Available formats are at least "TeX", "DVI" and "pdf" formats.

*kr@e-kr.org

In our notation, the torsion-free expressions of General Relativity are denoted with a hat on top, the metric signature is $(-,+,+,+)$ *and we consider units in which the speed of light* $c = 1$ *. Unless otherwise specified, Greek (Latin) letters denote spacetime (space) indices, bold letters denote tensor densities and round (square) brackets around indices denote symmetry (antisymmetry), as usual :*
$g_{(\mu\nu)} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})$, $g_{[\mu\nu]} = \frac{1}{2}(g_{\mu\nu} - g_{\nu\mu})$ *and* $\partial_\sigma \boldsymbol{g}^{[\mu\sigma]} = \frac{1}{2}\partial_\sigma(\sqrt{-g}(g^{\mu\sigma} - g^{\sigma\mu}))$ *with* $g = \det(g_{\mu\nu})$.

1 Motivation

In modern literature about non-symmetric theories of gravity such as the nonsymmetric gravitational theory [5], metric-affine theories of gravity [6] or supergravity [8], the transposition invariance is not imposed to the field equations. Albert Einstein introduced the requirement of transposition invariance in his last proposal for a relativistic theory of the non-symmetric field [1][2][3][4] and considered it as a fundamental principle serving to restrict the manifold of non-symmetric real expressions.

Plan of the paper. In section 2, we illustrate the requirement of transposition invariance by giving two important examples. In section 3, we derive the framework of non-symmetric $f(R)$ theories of gravity with non-symmetric metric and non-symmetric affine connection. In section 4, using the particular case of metric-affine $f(R)$ theories of gravity with symmetric metric and non-symmetric affine connection, we show that the requirement of transposition invariance leads to formally simpler and stronger field equations. In sections 5-6, we use the simplest metric-affine theory of gravity with torsion to physically reinterpret the ΛCDM cosmological by relating dark energy to the torsion of spacetime. In section 7, we show that the requirement of transposition invariance leads to formally simpler and stronger field equations for supergravity. In appendix A, we emphasize the consequences of imposing the transposition invariance to the metric-affine-vielbein formalism. In appendix B, we reply to common questions received about this paper.

2 The requirement of transposition invariance

The requirement of transposition invariance is a "weak condition of symmetry" that can be used to constrain non-symmetric real expressions (see [3] p128 or [4] p149).

A first example concerning the covariant derivative : if we replace $\mathbf{g}^{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ by their transposes $\mathbf{g}^{\nu\mu}$ and $\Gamma^\lambda_{\nu\mu}$ in the expression $\partial_\lambda \mathbf{g}^{\mu\nu} + \Gamma^\mu_{\sigma\lambda}\mathbf{g}^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma}\mathbf{g}^{\mu\sigma} - \Gamma^\sigma_{(\lambda\sigma)}\mathbf{g}^{\mu\nu}$, we obtain $\partial_\lambda \mathbf{g}^{\nu\mu} + \Gamma^\mu_{\lambda\sigma}\mathbf{g}^{\nu\sigma} + \Gamma^\nu_{\sigma\lambda}\mathbf{g}^{\sigma\mu} - \Gamma^\sigma_{(\lambda\sigma)}\mathbf{g}^{\nu\mu}$. If we interchange the free indices μ and ν in the latter, we get back the original expression $\partial_\lambda \mathbf{g}^{\mu\nu} + \Gamma^\mu_{\sigma\lambda}\mathbf{g}^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma}\mathbf{g}^{\mu\sigma} - \Gamma^\sigma_{(\lambda\sigma)}\mathbf{g}^{\mu\nu}$. One can see that this would not be the case with an expression such as $\partial_\lambda \mathbf{g}^{\mu\nu} + \Gamma^\mu_{\lambda\sigma}\mathbf{g}^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma}\mathbf{g}^{\mu\sigma} - \Gamma^\sigma_{(\lambda\sigma)}\mathbf{g}^{\mu\nu}$.

A second example concerning the curvature tensor : if we replace $\Gamma^\lambda_{\mu\nu}$ by its transpose $\Gamma^\lambda_{\nu\mu}$ in the expression $\partial_\sigma\Gamma^\sigma_{\mu\nu} - \frac{1}{2}\partial_\mu\Gamma^\sigma_{\sigma\nu} - \frac{1}{2}\partial_\nu\Gamma^\sigma_{\mu\sigma} - \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\sigma\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{(\sigma\tau)}$, we obtain $\partial_\sigma\Gamma^\sigma_{\nu\mu} - \frac{1}{2}\partial_\mu\Gamma^\sigma_{\nu\sigma} - \frac{1}{2}\partial_\nu\Gamma^\sigma_{\sigma\mu} - \Gamma^\sigma_{\tau\mu}\Gamma^\tau_{\nu\sigma} + \Gamma^\tau_{\nu\mu}\Gamma^\sigma_{(\sigma\tau)}$. If we interchange the free indices μ and ν in the latter, we get back the original expression $\partial_\sigma\Gamma^\sigma_{\mu\nu} - \frac{1}{2}\partial_\mu\Gamma^\sigma_{\sigma\nu} - \frac{1}{2}\partial_\nu\Gamma^\sigma_{\mu\sigma} - \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\sigma\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{(\sigma\tau)}$. One can see that this would not be the case with an expression such as $\partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\sigma\nu} - \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\sigma\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{\sigma\tau}$.

As explained in [1][2], imposing the transposition invariance to non-symmetric real expressions is equivalent to imposing the hermiticity to complex expressions (it would be similar to require $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ to be complex hermitian with respect to μ and ν). The requirement of transposition invariance is the cornerstone of this paper since it is systematically imposed to the non-symmetric field equations.

3 Non-symmetric $f(R)$ theories of gravity

We shall apply the method used in [2][3][4] to derive the field equations from the least action principle $\delta S = 0$ where $S = S_G(g_{\mu\nu}, \Gamma^\lambda_{\mu\nu}) + S_M(g_{\mu\nu}, \Gamma^\lambda_{\mu\nu}, \psi)$ where $S_G = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R)$ accounts for geometry and $S_M = \int d^4x \mathcal{L}_M$ accounts for matter (see [6] p8 for a detailed explanation). In order to derive transposition invariant field equations, the curvature scalar $R = g^{\mu\nu}R_{\mu\nu}$ shall be based on the curvature tensor $R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \frac{1}{2}\partial_\mu\Gamma^\sigma_{\sigma\nu} - \frac{1}{2}\partial_\nu\Gamma^\sigma_{\mu\sigma} - \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\sigma\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{(\sigma\tau)}$ introduced by Albert Einstein in [2].

Since the matter action S_M depends both on the metric and the affine connection, we have the stress-energy tensor $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}$ and the hypermomentum tensor $\Delta^{\mu\nu}_\lambda = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta\Gamma^\lambda_{\mu\nu}}$ (see [6](31) p9).

By variation of S with respect to the $g^{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$, we obtain (a prime denotes a derivation by R) :

$$\delta S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(K_{\mu\nu}\delta g^{\mu\nu} + M^{\mu\nu}_\lambda\delta\Gamma^\lambda_{\mu\nu}\right) + \frac{1}{2\kappa}\int d^4x\partial_\sigma\left(\sqrt{-g}T^\sigma\right)$$

where :

$$K_{\mu\nu} = f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \kappa T_{\mu\nu}$$

$$M^{\mu\nu}_\lambda = \frac{1}{\sqrt{-g}}\Big[-\nabla_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) + \frac{1}{2}\delta^\mu_\lambda\left(\nabla_\sigma\left(f'(R)\mathbf{g}^{\sigma\nu}\right) + \Gamma^\tau_{[\tau\sigma]}f'(R)\mathbf{g}^{\sigma\nu}\right) + \frac{1}{2}\delta^\nu_\lambda\left(\nabla_\sigma\left(f'(R)\mathbf{g}^{\mu\sigma}\right) + \Gamma^\tau_{[\sigma\tau]}f'(R)\mathbf{g}^{\mu\sigma}\right)\Big] - \kappa\Delta^{\mu\nu}_\lambda \qquad (1)$$

$$T^\sigma = g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - \frac{1}{2}g^{\sigma\nu}\delta\Gamma^\tau_{\tau\nu} - \frac{1}{2}g^{\mu\sigma}\delta\Gamma^\tau_{\mu\tau}$$

with :

$$R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \frac{1}{2}\partial_\mu\Gamma^\sigma_{\sigma\nu} - \frac{1}{2}\partial_\nu\Gamma^\sigma_{\mu\sigma} - \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\sigma\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{(\sigma\tau)} \text{ and } R = g^{\mu\nu}R_{\mu\nu}$$

$$\nabla_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) = \partial_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) + \Gamma^\mu_{\sigma\lambda}f'(R)\mathbf{g}^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma}f'(R)\mathbf{g}^{\mu\sigma} - \Gamma^\sigma_{(\lambda\sigma)}f'(R)\mathbf{g}^{\mu\nu}$$

$$\mathbf{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$$

The application of the least action principle leads to $\delta S = 0$. The last term of the expression of δS does not contribute to the integral since $\delta\Gamma^\lambda_{\mu\nu}$ vanish at the boundary of integration. Hence, we have the field equations $K_{\mu\nu} = 0$ and $M^{\mu\nu}_\lambda = 0$. Moreover, one can verify that $M^{[\mu\sigma]}_\sigma = 0$ implies $\frac{1}{\sqrt{-g}}\partial_\sigma(f'(R)\mathbf{g}^{[\mu\sigma]}) + 2\Gamma^\tau_{[\sigma\tau]}f'(R)g^{(\mu\sigma)} = 2\kappa\Delta^{[\mu\sigma]}_\sigma$. We therefore obtain the following field equations :

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\frac{1}{\sqrt{-g}}\Big[-\nabla_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) + \frac{1}{2}\delta^\mu_\lambda\left(\nabla_\sigma\left(f'(R)\mathbf{g}^{\sigma\nu}\right) + \Gamma^\tau_{[\tau\sigma]}f'(R)\mathbf{g}^{\sigma\nu}\right) + \frac{1}{2}\delta^\nu_\lambda\left(\nabla_\sigma\left(f'(R)\mathbf{g}^{\mu\sigma}\right) + \Gamma^\tau_{[\sigma\tau]}f'(R)\mathbf{g}^{\mu\sigma}\right)\Big] = \kappa\Delta^{\mu\nu}_\lambda$$

where :

$$R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \frac{1}{2}\partial_\mu\Gamma^\sigma_{\sigma\nu} - \frac{1}{2}\partial_\nu\Gamma^\sigma_{\mu\sigma} - \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\sigma\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{(\sigma\tau)} \text{ and } R = g^{\mu\nu}R_{\mu\nu} \qquad (2)$$

$$\nabla_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) = \partial_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) + \Gamma^\mu_{\sigma\lambda}f'(R)\mathbf{g}^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma}f'(R)\mathbf{g}^{\mu\sigma} - \Gamma^\sigma_{(\lambda\sigma)}f'(R)\mathbf{g}^{\mu\nu}$$

$$\mathbf{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$$

with : $\frac{1}{\sqrt{-g}}\partial_\sigma\left(f'(R)\mathbf{g}^{[\mu\sigma]}\right) + 2\Gamma^\tau_{[\sigma\tau]}f'(R)g^{(\mu\sigma)} = 2\kappa\Delta^{[\mu\sigma]}_\sigma$

The field equations (1)(2) are invariant under coordinates transformation and transposition invariant.

It is important to note that the field equations (1)(2) do not require $\Gamma^{\sigma}_{[\mu\sigma]} = 0$. In other words : the field equations are derived without having to constrain the torsion tensor by adding Lagrange multipliers to the action, which is not the case in [2] (see p734) and [6] (see (38) p10). In [2], two Lagrange multipliers are added to the action to obtain the field equations $\partial_\sigma \mathbf{g}^{[\mu\sigma]} = 0$ and $\Gamma^{\sigma}_{[\mu\sigma]} = 0$ (see p731-733 for a detailed explanation). In [6], a Lagrange multiplier is added to the action to obtain the field equation $\Gamma^{\sigma}_{[\mu\sigma]} = 0$ (see p9 for a detailed explanation).

4 Metric-affine $f(R)$ theories of gravity

The field equations of metric-affine $f(R)$ theories of gravity are a particular case of (1)(2) obtained by supposing that the metric tensor $g_{\mu\nu}$, the curvature tensor $R_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ are symmetric :

$$
\begin{aligned}
& f'(R)R_{\mu\nu} - \tfrac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} \\
& \tfrac{1}{\sqrt{-g}}\Big[-\nabla_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) + \tfrac{1}{2}\delta^{\mu}_{\lambda}\Big(\nabla_\sigma\left(f'(R)\mathbf{g}^{\sigma\nu}\right) + \Gamma^{\tau}_{[\tau\sigma]}f'(R)\mathbf{g}^{\sigma\nu}\Big) \\
& \qquad\qquad + \tfrac{1}{2}\delta^{\nu}_{\lambda}\Big(\nabla_\sigma\left(f'(R)\mathbf{g}^{\mu\sigma}\right) + \Gamma^{\tau}_{[\sigma\tau]}f'(R)\mathbf{g}^{\mu\sigma}\Big)\Big] = \kappa\Delta_{\lambda}^{\mu\nu} \\
& \text{where :} \\
& \quad R_{\mu\nu} = \partial_\sigma\Gamma^{\sigma}_{(\mu\nu)} - \tfrac{1}{2}\partial_\mu\Gamma^{\sigma}_{(\sigma\nu)} - \tfrac{1}{2}\partial_\nu\Gamma^{\sigma}_{(\mu\sigma)} - \Gamma^{\sigma}_{(\mu\tau)}\Gamma^{\tau}_{(\sigma\nu)} - \Gamma^{\sigma}_{[\mu\tau]}\Gamma^{\tau}_{[\sigma\nu]} + \Gamma^{\tau}_{(\mu\nu)}\Gamma^{\sigma}_{(\sigma\tau)} \text{ and } R = g^{\mu\nu}R_{\mu\nu} \\
& \quad \nabla_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) = \partial_\lambda\left(f'(R)\mathbf{g}^{\mu\nu}\right) + \Gamma^{\mu}_{\sigma\lambda}f'(R)\mathbf{g}^{\sigma\nu} + \Gamma^{\nu}_{\lambda\sigma}f'(R)\mathbf{g}^{\mu\sigma} - \Gamma^{\sigma}_{(\lambda\sigma)}f'(R)\mathbf{g}^{\mu\nu} \\
& \quad g_{\mu\nu} = g_{\nu\mu} \text{ , } \mathbf{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} \text{ , } R_{\mu\nu} = R_{\nu\mu} \text{ and } T_{\mu\nu} = T_{\nu\mu} \\
& \text{with : } \Gamma^{\tau}_{[\sigma\tau]}f'(R)g^{\mu\sigma} = \kappa\Delta_{\sigma}^{[\mu\sigma]}
\end{aligned}
\tag{3}
$$

These field equations are formally simpler (they do not require $\Gamma^{\sigma}_{[\mu\sigma]} = 0$) and stronger (they do not involve a Lagrange multiplier) than those derived in [6] (see (46)(47)(48) p10). This is due to the fact that they are derived from a transposition invariant action, which is not the case in [6] (the curvature tensor (41) used p10 to build the action is not transposition invariant).

There are other advantages coming from the requirement of transposition invariance. For instance, the affine connection $\Gamma^{\lambda}_{\mu\nu}$ becomes symmetric when the hypermomentum tensor $\Delta_{\lambda}^{\mu\nu}$ is symmetric : if we take the antisymmetric part of the second field equation of (3) when $\Delta_{\lambda}^{[\mu\nu]} = 0$, we have $\Gamma^{\mu}_{[\lambda\sigma]}g^{\sigma\nu} + \Gamma^{\nu}_{[\sigma\lambda]}g^{\mu\sigma} = 0$ which leads to $\Gamma^{\lambda}_{[\mu\nu]} = 0$. In particular, this means that (3) are the field equations of General Relativity when $f(R) = R$ and $\Delta_{\lambda}^{\mu\nu} = 0$.

We will further simplify the field equations (3) by imposing the metricity to the symmetric part $\Gamma^{\lambda}_{(\mu\nu)}$ of the affine connection. This can be done by supposing that the hypermomentum tensor is antisymmetric : when $\Delta_{\lambda}^{\mu\nu} = -\Delta_{\lambda}^{\nu\mu}$, the second equation becomes $\frac{1}{\sqrt{-g}}[\nabla_\lambda(f'(R)\mathbf{g}^{\mu\nu})] = -\kappa\Delta_{\lambda}^{[\mu\nu]}$ whose symmetric part leads to the equation $\partial_\lambda(f'(R)g^{\mu\nu}) + \Gamma^{\mu}_{(\sigma\lambda)}f'(R)g^{\sigma\nu} + \Gamma^{\nu}_{(\lambda\sigma)}f'(R)g^{\mu\sigma} = 0$ having the solution $\Gamma^{\lambda}_{(\mu\nu)} = \hat{\Gamma}^{\lambda}_{\mu\nu} + \frac{1}{2f'(R)}[\delta^{\lambda}_{\mu}\partial_\nu f'(R) + \delta^{\lambda}_{\nu}\partial_\mu f'(R) - g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma f'(R)]$ with the Christoffel symbols $\hat{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$; and whose antisymmetric part leads to the equation $\Gamma^{\mu}_{[\lambda\sigma]}f'(R)g^{\sigma\nu} + \Gamma^{\nu}_{[\sigma\lambda]}f'(R)g^{\mu\sigma} = \kappa\Delta_{\lambda}^{[\mu\nu]}$ having the solution $\Gamma_{\lambda[\mu\nu]} = \frac{1}{2}\frac{\kappa}{f'}(\Delta_{\lambda[\mu\nu]} + \Delta_{\mu[\lambda\nu]} + \Delta_{\nu[\mu\lambda]})$.

Therefore, when $\Delta_\lambda^{\mu\nu} = \Delta_\lambda^{[\mu\nu]}$, the metric-affine field equations (3) become :

$$\left.\begin{array}{l}
f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} \\
\Gamma^\mu_{[\lambda\sigma]}f'(R)g^{\sigma\nu} + \Gamma^\nu_{[\sigma\lambda]}f'(R)g^{\mu\sigma} = \kappa\Delta_\lambda^{[\mu\nu]} \;\Rightarrow\; \Gamma_{\lambda[\mu\nu]} = \frac{\kappa}{2f'(R)}\left(\Delta_{\lambda[\mu\nu]} + \Delta_{\mu[\lambda\nu]} + \Delta_{\nu[\mu\lambda]}\right) \\
\text{where :} \\
\quad R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{(\mu\nu)} - \frac{1}{2}\partial_\mu\Gamma^\sigma_{(\sigma\nu)} - \frac{1}{2}\partial_\nu\Gamma^\sigma_{(\mu\sigma)} - \Gamma^\sigma_{(\mu\tau)}\Gamma^\tau_{(\sigma\nu)} - \Gamma^\sigma_{[\mu\tau]}\Gamma^\tau_{[\sigma\nu]} + \Gamma^\tau_{(\mu\nu)}\Gamma^\sigma_{(\sigma\tau)} \text{ and } R = g^{\mu\nu}R_{\mu\nu} \\
\quad \Gamma^\lambda_{(\mu\nu)} = \hat{\Gamma}^\lambda_{\mu\nu} + \frac{1}{2f'(R)}\left[\delta^\lambda_\mu\partial_\nu f'(R) + \delta^\lambda_\nu\partial_\mu f'(R) - g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma f'(R)\right] \\
\quad \hat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\
\quad g_{\mu\nu} = g_{\nu\mu} \text{ , } R_{\mu\nu} = R_{\nu\mu} \text{ and } T_{\mu\nu} = T_{\nu\mu} \\
\text{with : } \Gamma^\tau_{[\sigma\tau]}f'(R)g^{\mu\sigma} = \kappa\Delta_\sigma^{[\mu\sigma]}
\end{array}\right| \quad (4)$$

These field equations are easy to understand from a physical point of view : matter interacts with geometry through the symmetric stress-energy tensor $T_{\mu\nu}$ and/or through the antisymmetric hypermomentum tensor $\Delta_\lambda^{[\mu\nu]}$. Only the hypermomentum tensor can generate torsion and this torsion does not propagate. In other words : torsion can only exist in presence of matter (energy).

5 The modified Friedmann equations

In this section, we find a cosmological solution to the field equations (4). We shall use the Friedmann-Lemaitre-Robertson-Walker metric without spatial curvature $g_{00} = -1$ and $g_{ij} = a^2\delta_{ij}$ with $a = a(t) \Rightarrow g^{00} = -1$, $g^{ij} = \frac{1}{a^2}\delta_{ij}$ and $\sqrt{-g} = a^3$ (see [7] (7) p6). We shall also use the standard stress-energy tensor $T_{00} = \rho$ and $T_{ij} = a^2P\delta_{ij}$ (see [7] (16) p8).

We will suppose that the hypermomentum tensor is given by $\Delta_\lambda^{[\mu\nu]} = (\delta^\mu_\lambda g^{\alpha\nu} - \delta^\nu_\lambda g^{\mu\alpha})A_\alpha + \frac{\epsilon^{\sigma\mu\nu\alpha}}{\sqrt{-g}}g_{\sigma\lambda}B_\alpha$ where A_α and B_α are two hypothetical material vector fields and where $\epsilon^{\mu\nu\sigma\alpha}$ is the Levi-Civita tensor density. Taking into account the homogeneity and the isotropy of spacetime, only the time-components ($A_0 = A$ and $B_0 = B$) of the two material vector fields do not vanish to give $\Delta_k^{[0j]} = -\Delta_k^{[j0]} = A\delta_{jk}$ and $\Delta_k^{[ij]} = -\Delta_k^{[ji]} = \frac{B}{a}\epsilon_{ijk}$ (with $\epsilon_{123} = 1$ and $\epsilon_{imn}\epsilon_{jmn} = 2\delta_{ij}$) . From (4), we have (all other components vanishing) :

$$\Gamma^0_{00} = \frac{1}{2}\frac{\dot{f}'}{f'}$$

$$\Gamma^k_{0j} = (h + \frac{1}{2}\frac{\dot{f}'}{f'} - \frac{\kappa A}{f'})\delta_{jk}$$

$$\Gamma^k_{j0} = (h + \frac{1}{2}\frac{\dot{f}'}{f'} + \frac{\kappa A}{f'})\delta_{jk}$$

$$\Gamma^0_{ij} = \Gamma^0_{ji} = a^2(h + \frac{1}{2}\frac{\dot{f}'}{f'})\delta_{ij}$$

$$\Gamma^k_{ij} = -\Gamma^k_{ji} = \frac{1}{2}\frac{\kappa a B}{f'}\epsilon_{ijk}$$

$$\Gamma^\sigma_{[0\sigma]} = -3\frac{\kappa A}{f'} \text{ , } \Gamma^\sigma_{(0\sigma)} = 3h + 2\frac{\dot{f}'}{f'} \text{ , } \Gamma^\sigma_{[i\sigma]} = 0 \text{ , } \Gamma^\sigma_{(i\sigma)} = 0$$

with $h = \frac{\dot{a}}{a}$

In the previous relations, a prime denotes derivative with respect to R and a dot denotes derivative with respect to the time t .

From (4), we obtain :

$$R_{00} = -3\frac{\ddot{a}}{a} - \frac{3}{2}\frac{\ddot{f}'}{f'} - \frac{3}{2}h\frac{\dot{f}'}{f'} + \frac{3}{2}\frac{\dot{f}'}{f'}\frac{\dot{f}'}{f'} + 3\frac{\kappa^2 A^2}{f'f'}$$
$$R_{ij} = a^2\left[\frac{\ddot{a}}{a} + 2h^2 + \frac{1}{2}\frac{\ddot{f}'}{f'} + \frac{5}{2}h\frac{\dot{f}'}{f'} - \frac{1}{2}\frac{\kappa^2 B^2}{f'f'}\right]\delta_{ij}$$
$$\text{with } h = \frac{\dot{a}}{a} \Rightarrow \dot{h} = \frac{\ddot{a}}{a} - h^2$$

The field equation $f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu}$ gives :

$$-3f'\frac{\ddot{a}}{a} - \frac{3}{2}\ddot{f}' - \frac{3}{2}h\dot{f}' + \frac{3}{2}\frac{\dot{f}'\dot{f}'}{f'} + 3\frac{\kappa^2 A^2}{f'} + \frac{1}{2}f = \kappa\rho$$
$$f'\frac{\ddot{a}}{a} + 2f'h^2 + \frac{1}{2}\ddot{f}' + \frac{5}{2}h\dot{f}' - \frac{1}{2}\frac{\kappa^2 B^2}{f'} - \frac{1}{2}f = \kappa P$$

Eliminating $\frac{\ddot{a}}{a}$ from these relations leads to the first modified Friedmann equation :

$$\left(h + \frac{1}{2}\frac{\dot{f}'}{f'}\right)^2 = \frac{1}{6}\frac{(\kappa\rho + 3\kappa P + f)}{f'} - \frac{1}{2}\frac{\kappa^2 A^2}{f'f'} + \frac{1}{4}\frac{\kappa^2 B^2}{f'f'} \tag{5}$$

We have $R_{00} = -3\frac{\ddot{a}}{a} - \frac{3}{2}\frac{\ddot{f}'}{f'} - \frac{3}{2}h\frac{\dot{f}'}{f'} + \frac{3}{2}\frac{\dot{f}'}{f'}\frac{\dot{f}'}{f'} + 3\frac{\kappa^2 A^2}{f'f'}$ and also $f'R_{00} + \frac{1}{2}f = \kappa\rho$. Eliminating R_{00} from these relations leads to the second modified Friedmann equation :

$$\frac{\ddot{a}}{a} = \frac{1}{6}\frac{(f - 2\kappa\rho)}{f'} - \frac{1}{2}\frac{\ddot{f}'}{f'} - \frac{1}{2}h\frac{\dot{f}'}{f'} + \frac{1}{2}\frac{\dot{f}'}{f'}\frac{\dot{f}'}{f'} + \frac{\kappa^2 A^2}{f'f'} \tag{6}$$

The ΛCDM model (see [7] (29) p10) is the simplest cosmological model that is in general agreement with the observations, in particular with the acceleration of the universe. The field equations of the ΛCDM model are $h^2 = \frac{1}{3}\kappa\rho + \frac{1}{3}\Lambda$ and $\frac{\ddot{a}}{a} = -\frac{1}{6}\kappa\rho + \frac{1}{3}\Lambda$. The simplest way to obtain these equations from (5) and (6) is to set $P = 0$, $f = R \Rightarrow R = \kappa\rho$, $\kappa^2 A^2 = \frac{1}{3}\Lambda$ and $\kappa^2 B^2 = 2\Lambda$. From the observations (see [7], (64) p25), we have $\frac{\kappa\rho_o}{3h_o^2} = 0.3$ and $\frac{\Lambda}{3h_o^2} = 0.7$ where the indice "o" denotes the value of the parameter at the present time. We therefore obtain $\frac{\kappa A}{3h_o} = 0.278$ and $\frac{\kappa B}{3h_o} = 0.683$.

6 The ΛCDM model revisited

In this section, we deduce the ΛCDM model from a more rigorous point of view. When $f(R) = R$, the field equations (4) lead to the simplest metric-affine theory of gravity with torsion :

$$\left.\begin{array}{l}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \;\Rightarrow\; \hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} - \Gamma^{\tau}_{[\mu\sigma]}\Gamma^{\sigma}_{[\tau\nu]} + \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\Gamma^{\tau}_{[\alpha\sigma]}\Gamma^{\sigma}_{[\tau\beta]} = \kappa T_{\mu\nu} \\
\Gamma^{\mu}_{[\lambda\sigma]}g^{\sigma\nu} + \Gamma^{\nu}_{[\sigma\lambda]}g^{\mu\sigma} = \kappa\Delta_{\lambda}^{[\mu\nu]} \;\Rightarrow\; \Gamma_{\lambda[\mu\nu]} = \frac{1}{2}\kappa\left(\Delta_{\lambda[\mu\nu]} + \Delta_{\mu[\lambda\nu]} + \Delta_{\nu[\mu\lambda]}\right) \\
\text{where :} \\
\quad R_{\mu\nu} = \hat{R}_{\mu\nu} - \Gamma^{\tau}_{[\mu\sigma]}\Gamma^{\sigma}_{[\tau\nu]} \text{ and } R = g^{\mu\nu}R_{\mu\nu} \Rightarrow R = g^{\mu\nu}\hat{R}_{\mu\nu} - g^{\mu\nu}\Gamma^{\tau}_{[\mu\sigma]}\Gamma^{\sigma}_{[\tau\nu]} = \hat{R} - g^{\mu\nu}\Gamma^{\tau}_{[\mu\sigma]}\Gamma^{\sigma}_{[\tau\nu]} \\
\quad \hat{R}_{\mu\nu} = \partial_\sigma\hat{\Gamma}^{\sigma}_{\mu\nu} - \partial_\mu\hat{\Gamma}^{\sigma}_{\sigma\nu} - \hat{\Gamma}^{\sigma}_{\mu\tau}\hat{\Gamma}^{\tau}_{\sigma\nu} + \hat{\Gamma}^{\tau}_{\mu\nu}\hat{\Gamma}^{\sigma}_{\sigma\tau} \text{ and } \hat{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\
\quad g_{\mu\nu} = g_{\nu\mu} \text{ , } R_{\mu\nu} = R_{\nu\mu} \text{ and } T_{\mu\nu} = T_{\nu\mu} \\
\text{with : } \Gamma^{\tau}_{[\sigma\tau]}g^{\mu\sigma} = \kappa\Delta_{\sigma}^{[\mu\sigma]}
\end{array}\right| \tag{7}$$

It is easy to verify that these field equations imply the transposition invariant metric-affine equation $\partial_\lambda g_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda}g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu}g_{\mu\sigma} = -\Delta_{\lambda[\mu\nu]}$.

As in the previous section, we will suppose that the hypermomentum tensor is given by $\Delta_{\lambda}^{[\mu\nu]} = (\delta^{\mu}_{\lambda} g^{\alpha\nu} - \delta^{\nu}_{\lambda} g^{\mu\alpha}) A_{\alpha} + \frac{\epsilon^{\sigma\mu\nu\alpha}}{\sqrt{-g}} g_{\sigma\lambda} B_{\alpha}$ where A_{α} and B_{α} are two hypothetical material vector fields and where $\epsilon^{\mu\nu\sigma\alpha}$ is the Levi-Civita tensor density with $\epsilon^{0123} = -1$, $\epsilon_{0123} = 1$, $\epsilon^{\mu\nu\lambda\tau} = -g\epsilon_{\alpha\beta\sigma\theta} g^{\alpha\mu} g^{\beta\nu} g^{\sigma\lambda} g^{\theta\tau}$ and $\epsilon^{\mu\tau\sigma\alpha}\epsilon_{\nu\lambda\sigma\alpha} = -2\delta^{\mu}_{\nu}\delta^{\tau}_{\lambda} + 2\delta^{\mu}_{\lambda}\delta^{\tau}_{\nu} \Rightarrow \epsilon^{\mu\tau\sigma\alpha}\epsilon_{\nu\tau\sigma\alpha} = -6\delta^{\mu}_{\nu}$.

The second equation of (7) gives $\Gamma^{\lambda}_{[\mu\nu]} = \kappa(\delta^{\lambda}_{\mu} A_{\nu} - \delta^{\lambda}_{\nu} A_{\mu} - \frac{1}{2}\sqrt{-g}\epsilon_{\sigma\mu\nu\alpha} g^{\sigma\lambda} B^{\alpha}) \Rightarrow \Gamma^{\tau}_{[\mu\sigma]}\Gamma^{\sigma}_{[\tau\nu]} = \kappa^2(-3A_{\mu}A_{\nu} + \frac{1}{2}B_{\mu}B_{\nu} - \frac{1}{2}g_{\mu\nu}B_{\sigma}B^{\sigma})$. Hence, we obtain from the first equation $\hat{G}_{\mu\nu} \stackrel{\text{def}}{=} \hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} = \kappa T_{\mu\nu} + \kappa^2(-3A_{\mu}A_{\nu} + \frac{1}{2}B_{\mu}B_{\nu} + \frac{3}{2}g_{\mu\nu}A_{\sigma}A^{\sigma} + \frac{1}{4}g_{\mu\nu}B_{\sigma}B^{\sigma})$. By setting $B_{\mu}B_{\nu} = 6A_{\mu}A_{\nu}$, we can simplify this equation to obtain $\hat{G}_{\mu\nu} = \kappa T_{\mu\nu} + 3\kappa^2 g_{\mu\nu}A_{\sigma}A^{\sigma}$. From General Relativity we have the equation $\hat{\nabla}_{\tau}\hat{G}^{\mu\tau} = 0$ and since dark energy is supposed not to interact with ordinary matter, this means that we must have $\hat{\nabla}_{\tau}T^{\mu\tau} = 0$ and $\hat{\nabla}_{\tau}(g^{\mu\tau}A_{\sigma}A^{\sigma}) = 0 \Rightarrow A_{\sigma}A^{\sigma}$ constant. If we set $A_{\sigma}A^{\sigma} = -\frac{1}{3\kappa^2}\Lambda$, we obtain the field equation of General Relativity $\hat{G}_{\mu\nu} = \kappa T_{\mu\nu} - g_{\mu\nu}\Lambda$ with the cosmological constant Λ .

The expressions we chose for the hypermomentum tensor components are certainly questionable. Nevertheless, it shows that the cosmological constant emerges from the hypermomentum tensor components, which could explain its observed smallness if only few particle fields contribute to the hypermomentum tensor to generate torsion.

7 $N = 1$ pure supergravity revisited

In our notation, the torsion-free expressions of supergravity are denoted with a hat on top, the metric signature is $(-,+,+,+)$, Greek (Latin) letters denote covariant (Lorentz) indices and for convenience, we consider units in which $c = \kappa = 1$.

Using the non-transposition invariant metric-affine equation $\partial_{\lambda}g_{\mu\nu} - \Gamma^{\sigma}{}_{\lambda\mu}g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu}g_{\mu\sigma} = 0$, the field equations of $N = 1$ pure supergravity are (see [8] and references therein) :

$$\left|\begin{array}{l} \delta S = \delta\left[\frac{1}{2}\int d^4x e e^{a\mu} e^{b\nu} R_{\mu\nu ab} - \frac{1}{2}\int d^4x e \bar{\psi}_{\mu}\gamma^{\mu\nu\sigma} D_{\nu}\psi_{\sigma}\right] = 0 \\ \text{where :} \\ \quad R_{\mu\nu ab} = \partial_{\mu}\omega_{\nu ab} - \partial_{\nu}\omega_{\mu ab} + \omega_{\mu ac}\omega_{\nu}{}^{c}{}_{b} - \omega_{\nu ac}\omega_{\mu}{}^{c}{}_{b} \\ \quad D_{\nu}\psi_{\sigma} = \partial_{\nu}\psi_{\sigma} + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\psi_{\sigma} \\ \quad \delta e_{\mu}{}^{a} = \frac{1}{2}\bar{\epsilon}\gamma^{a}\psi_{\mu} \;\text{ and }\; \delta\psi_{\mu} = D_{\mu}\epsilon = \partial_{\mu}\epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon \\ \text{with :} \\ \quad \omega_{\lambda\mu\nu} = \omega_{\lambda[\mu\nu]} = \hat{\omega}_{\lambda[\mu\nu]} + K_{\lambda[\mu\nu]} \\ \quad \hat{\omega}_{\lambda[ab]} = \frac{1}{2}e_{a}{}^{\sigma}(\partial_{\lambda}e_{b\sigma} - \partial_{\sigma}e_{b\lambda}) - \frac{1}{2}e_{b}{}^{\sigma}(\partial_{\lambda}e_{a\sigma} - \partial_{\sigma}e_{a\lambda}) - \frac{1}{2}e_{a}{}^{\sigma}e_{b}{}^{\tau}(\partial_{\sigma}e_{\tau}{}^{c} - \partial_{\tau}e_{\sigma}{}^{c})e_{c\lambda} \\ \quad K_{\lambda[\mu\nu]} = \frac{1}{2}(T_{\lambda[\mu\nu]} + T_{\mu[\lambda\nu]} + T_{\nu[\mu\lambda]}) \;\text{ and }\; \Gamma^{\lambda}{}_{[\mu\nu]} = K_{[\mu}{}^{\lambda}{}_{\nu]} = \frac{1}{2}T^{\lambda}{}_{[\mu\nu]} \end{array}\right| \tag{8}$$

In these equations, $T^{\lambda}{}_{[\mu\nu]}$ is the Cartan tensor and $K^{\lambda}{}_{[\mu\nu]}$ is the contortion tensor (see appendix A). In the so-called 1.5 order formalism (see [8] p5), the field equations (8) imply $T^{\lambda}{}_{[\mu\nu]} = \frac{1}{2}\bar{\psi}_{\mu}\gamma^{\lambda}\psi_{\nu}$. The use of the Lorentz derivative D_{ν} to define the action S is questionable but justified by the fact that using the covariant derivate ∇_{ν} would not lead to supersymmetric field equations.

When imposing the transposition invariance to supergravity, it is shown in appendix A that the metric-affine equation becomes $\partial_\lambda g_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma} = -\Delta_{\lambda[\mu\nu]}$, with $\Gamma^\lambda{}_{[\mu\nu]}$ playing the role of $K^\lambda{}_{[\mu\nu]}$ and $\Delta^\lambda{}_{[\mu\nu]}$ playing the role of $T^\lambda{}_{[\mu\nu]}$ in the field equations (8). Therefore, imposing the transposition invariance to $N = 1$ pure supergravity leads to the following field equations :

$$
\left|\begin{array}{l}
\delta S = \delta\left[\frac{1}{2}\int d^4x e e^{a\mu} e^{b\nu} R_{\mu\nu ab} - \frac{1}{2}\int d^4x e \bar{\psi}_\mu \gamma^{\mu\nu\sigma} D_\nu \psi_\sigma\right] = 0 \\
\text{where :} \\
\quad R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b \\
\quad D_\nu \psi_\sigma = \partial_\nu \psi_\sigma + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\psi_\sigma \\
\quad \delta e_\mu{}^a = \frac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu \;\text{ and }\; \delta\psi_\mu = D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon \\
\text{with :} \\
\quad \omega_{\lambda\mu\nu} = \omega_{\lambda[\mu\nu]} = \hat{\omega}_{\lambda[\mu\nu]} + \Gamma_{\lambda[\mu\nu]} \\
\quad \hat{\omega}_{\lambda[ab]} = \frac{1}{2}e_a{}^\sigma(\partial_\lambda e_{b\sigma} - \partial_\sigma e_{b\lambda}) - \frac{1}{2}e_b{}^\sigma(\partial_\lambda e_{a\sigma} - \partial_\sigma e_{a\lambda}) - \frac{1}{2}e_a{}^\sigma e_b{}^\tau(\partial_\sigma e_\tau{}^c - \partial_\tau e_\sigma{}^c)e_{c\lambda} \\
\quad \Gamma_{\lambda[\mu\nu]} = \frac{1}{2}(\Delta_{\lambda[\mu\nu]} + \Delta_{\mu[\lambda\nu]} + \Delta_{\nu[\mu\lambda]})
\end{array}\right. \tag{9}
$$

These field equations are formally simpler (the use of the contortion tensor $K^\lambda{}_{[\mu\nu]}$ is avoided) and stronger (the use of the Lorentz derivative D_μ is justified by the fact that the use of the covariant derivative ∇_μ is no longer necessary, as explained in appendix A) than the field equations (8).

In the so-called 1.5 order formalism (see [8] p5), the field equations (9) imply $\Delta^\lambda{}_{[\mu\nu]} = \frac{1}{2}\bar{\psi}_\mu \gamma^\lambda \psi_\nu$. As we saw in the previous section, the fact of introducing the hypermomentum tensor $\Delta^\lambda{}_{[\mu\nu]}$ could allow to include dark energy in supergravity through material fields.

It is important to note that the field equations (8) and (9) are the same when interpreted in terms of torsion-free connections : following [8](23) p6, the action S of (8) and (9) can be expended into $S = \frac{1}{2}\int d^4x e[\hat{R} - \bar{\psi}_\mu \gamma^{\mu\nu\sigma}\hat{D}_\nu \psi_\sigma + \mathcal{L}_{torsion}]$ with $\mathcal{L}_{torsion} = -K_{\lambda\mu\nu}K^{\nu\mu\lambda} - K^\sigma{}_{\mu\sigma}K^\epsilon{}_\epsilon{}^\mu = g^{\mu\nu}(\Gamma^\tau{}_{[\mu\sigma]}\Gamma^\sigma{}_{[\tau\nu]} - \Gamma^\sigma{}_{[\mu\sigma]}\Gamma^\tau{}_{[\tau\nu]}) = -\frac{1}{16}(\bar{\psi}^\mu \gamma^\lambda \psi^\nu)(\bar{\psi}_\mu \gamma_\lambda \psi_\nu + 2\bar{\psi}_\mu \gamma_\nu \psi_\lambda) + \frac{1}{4}(\bar{\psi}_\mu \gamma_\sigma \psi^\sigma)(\bar{\psi}^\mu \gamma_\tau \psi^\tau)$. In other words, it is only when (8) and (9) are interpreted in terms of connections with torsion that they differ.

8 Conclusion

The advantage of imposing the transposition invariance to metric-affine theories of gravity is to derive field equations without having to constrain the torsion tensor. Another advantage is the possibility of relating dark energy to the torsion of spacetime through the hypermomentum tensor, which could explain the smallness of the observed cosmological constant if only few particle fields contribute to the hypermomentum tensor to generate torsion.

The advantage of imposing the transposition invariance to supergravity is to avoid the use of the contortion tensor and to justify the use of the Lorentz derivative to derive the field equations. Another advantage would be the inclusion of dark energy through the hypermomentum tensor.

Appendix A

In this appendix, we emphasize the consequences of imposing the transposition invariance to the metric-affine-vielbein formalism with $f(R) = R$, $g_{\mu\nu} = g_{\nu\mu}$ and $\Delta^{\lambda}{}_{\mu\nu} = \Delta^{\lambda}{}_{[\mu\nu]}$. *In our notation, the torsion-free expressions of General Relativity and supergravity are denoted with a hat on top, the metric signature is* $(-,+,+,+)$ *, Greek (Latin) letters denote covariant (Lorentz) indices and for convenience, we consider units in which* $c = \kappa = 1$ *.*

The vielbein $e_{\mu}{}^{a}$ or $e_{a}{}^{\mu}$ is defined by the relations $e_{\mu}{}^{a} e_{\nu}{}^{b} \eta_{ab} = g_{\mu\nu}$ or $e_{a}{}^{\mu} e_{b}{}^{\nu} g_{\mu\nu} = \eta_{ab}$ where η_{ab} is the Lorentz metric tensor $(-1, 1, 1, 1)$. These relations imply that $e = \det(e_{\mu}{}^{a}) = \sqrt{-g}$, $e_{\mu}{}^{a} e_{a}{}^{\nu} = \delta^{\nu}_{\mu}$ and $e_{a}{}^{\mu} e_{\mu}{}^{b} = \delta^{b}_{a}$. The Lorentz derivative is defined by $D_{\mu} e_{\nu}{}^{a} = \partial_{\mu} e_{\nu}{}^{a} + \omega_{\mu}{}^{a}{}_{c} e_{\nu}{}^{c}$ with the spin connection $\omega_{\mu ab}$ specified so that $D_{\mu}\eta_{ab} = \partial_{\mu}\eta_{ab} - \omega_{\mu}{}^{c}{}_{a}\eta_{cb} - \omega_{\mu}{}^{c}{}_{b}\eta_{ac} = 0 \Rightarrow \omega_{\mu ab} = \omega_{\mu[ab]}$.

The metric-affine equation behind (8) is $\partial_{\lambda} g_{\mu\nu} - \Gamma^{\sigma}{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} g_{\mu\sigma} = 0$. This equation is not transposition invariant and does not imply $\partial_{\lambda} g_{\mu\nu} - \Gamma^{\sigma}{}_{(\lambda\mu)} g_{\sigma\nu} - \Gamma^{\sigma}{}_{(\lambda\nu)} g_{\mu\sigma} = 0$ when $\Gamma^{\lambda}{}_{[\mu\nu]} \neq 0$. Its solution is $\Gamma^{\lambda}{}_{\mu\nu} = \hat{\Gamma}^{\lambda}{}_{\mu\nu} + K_{\mu}{}^{\lambda}{}_{\nu}$ with the Christoffel symbols $\hat{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu})$ and the contortion tensor $K_{\lambda\mu\nu} = K_{\lambda[\mu\nu]} = \frac{1}{2}(T_{\lambda[\mu\nu]} + T_{\mu[\lambda\nu]} + T_{\nu[\mu\lambda]})$ which is based on the Cartan tensor $T^{\lambda}{}_{[\mu\nu]}$. One can see from these relations that $\Gamma^{\lambda}{}_{[\mu\nu]} = K_{[\mu}{}^{\lambda}{}_{\nu]} = \frac{1}{2} T^{\lambda}{}_{[\mu\nu]}$.

The transposition invariant metric-affine expression behind (9) is $\partial_{\lambda} g_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} g_{\mu\sigma}$, which does no longer allow to introduce the torsion tensor through the Cartan tensor. Following the second equation of (7), the simplest way to introduce the torsion tensor $\Gamma^{\lambda}{}_{[\mu\nu]}$ is through the hypermomentum tensor $\Delta^{\lambda}{}_{[\mu\nu]}$ defined by the field equation $\partial_{\lambda} g_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} g_{\mu\sigma} = -\Delta_{\lambda[\mu\nu]}$ whose symmetric part leads to the equation $\hat{\nabla}_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \hat{\Gamma}^{\sigma}{}_{\mu\lambda} g_{\sigma\nu} - \hat{\Gamma}^{\sigma}{}_{\lambda\nu} g_{\mu\sigma} = 0$ with the Christoffel symbols $\hat{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu})$; and whose antisymmetric part leads to the equation $\Gamma_{\mu[\lambda\nu]} + \Gamma_{\nu[\mu\lambda]} = \Delta_{\lambda[\mu\nu]}$ with $\Gamma_{\lambda[\mu\nu]} = \frac{1}{2}(\Delta_{\lambda[\mu\nu]} + \Delta_{\mu[\lambda\nu]} + \Delta_{\nu[\mu\lambda]})$.

The vielbein postulate behind (8) is $\nabla_{\mu} e_{\nu}{}^{a} = D_{\mu} e_{\nu}{}^{a} - \Gamma^{\sigma}{}_{\mu\nu} e_{\sigma}{}^{a} = \partial_{\mu} e_{\nu}{}^{a} + \omega_{\mu}{}^{a}{}_{c} e_{\nu}{}^{c} - \Gamma^{\sigma}{}_{\mu\nu} e_{\sigma}{}^{a} = 0$, which leads to the equations $\partial_{\lambda} g_{\mu\nu} - \Gamma^{\sigma}{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} g_{\mu\sigma} = 0$ and $D_{\mu} e_{\nu a} - D_{\nu} e_{\mu a} = T_{a[\mu\nu]}$. The latter allows to deduce the expressions $\hat{\omega}_{\lambda[ab]} = \frac{1}{2} e_{a}{}^{\sigma}(\partial_{\lambda} e_{b\sigma} - \partial_{\sigma} e_{b\lambda}) - \frac{1}{2} e_{b}{}^{\sigma}(\partial_{\lambda} e_{a\sigma} - \partial_{\sigma} e_{a\lambda}) - \frac{1}{2} e_{a}{}^{\sigma} e_{b}{}^{\tau}(\partial_{\sigma} e_{\tau}{}^{c} - \partial_{\tau} e_{\sigma}{}^{c}) e_{c\lambda}$ and $K_{\lambda[\mu\nu]} = \frac{1}{2}(T_{\lambda[\mu\nu]} + T_{\mu[\lambda\nu]} + T_{\nu[\mu\lambda]})$ for $\omega_{\lambda[ab]} = \hat{\omega}_{\lambda[ab]} + K_{\lambda[ab]}$.

When imposing the transposition invariance, we saw above that the simplest way to introduce the torsion tensor $\Gamma^{\lambda}{}_{[\mu\nu]}$ is through the hypermomentum tensor $\Delta^{\lambda}{}_{[\mu\nu]}$ defined by the field equation $\partial_{\lambda} g_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu} g_{\mu\sigma} = -\Delta_{\lambda[\mu\nu]}$ whose symmetric part leads to the equation $\hat{\nabla}_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \hat{\Gamma}^{\sigma}{}_{\mu\lambda} g_{\sigma\nu} - \hat{\Gamma}^{\sigma}{}_{\lambda\nu} g_{\mu\sigma} = 0$ and whose antisymmetric part leads to the equation $\Gamma_{\mu[\lambda\nu]} + \Gamma_{\nu[\mu\lambda]} = \Delta_{\lambda[\mu\nu]}$. Hence, the vielbein postulate behind (9) is $\hat{\nabla}_{\mu} e_{\nu}{}^{a} = \hat{D}_{\mu} e_{\nu}{}^{a} - \hat{\Gamma}^{\sigma}{}_{\mu\nu} e_{\sigma}{}^{a} = \partial_{\mu} e_{\nu}{}^{a} + \hat{\omega}_{\mu}{}^{a}{}_{c} e_{\nu}{}^{c} - \hat{\Gamma}^{\sigma}{}_{\mu\nu} e_{\sigma}{}^{a} = 0$, which leads to the equation $\hat{\nabla}_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \hat{\Gamma}^{\sigma}{}_{\mu\lambda} g_{\sigma\nu} - \hat{\Gamma}^{\sigma}{}_{\lambda\nu} g_{\mu\sigma} = 0$. Also, the use of the covariant derivative ∇_{μ} is no longer necessary since the use of the Lorentz derivative $D_{\mu} e_{\nu}{}^{a} = \partial_{\mu} e_{\nu}{}^{a} + \omega_{\mu}{}^{a}{}_{c} e_{\nu}{}^{c}$ is sufficient to obtain the equation $D_{\mu} e_{\nu a} - D_{\nu} e_{\mu a} = \Gamma_{\mu[a\nu]} - \Gamma_{\nu[a\mu]} = \Delta_{a[\mu\nu]}$ when $\omega_{\lambda[ab]} = \hat{\omega}_{\lambda[ab]} + \Gamma_{\lambda[ab]}$. The latter allows to deduce the expressions $\hat{\omega}_{\lambda[ab]} = \frac{1}{2} e_{a}{}^{\sigma}(\partial_{\lambda} e_{b\sigma} - \partial_{\sigma} e_{b\lambda}) - \frac{1}{2} e_{b}{}^{\sigma}(\partial_{\lambda} e_{a\sigma} - \partial_{\sigma} e_{a\lambda}) - \frac{1}{2} e_{a}{}^{\sigma} e_{b}{}^{\tau}(\partial_{\sigma} e_{\tau}{}^{c} - \partial_{\tau} e_{\sigma}{}^{c}) e_{c\lambda}$ and $\Gamma_{\lambda[\mu\nu]} = \frac{1}{2}(\Delta_{\lambda[\mu\nu]} + \Delta_{\mu[\lambda\nu]} + \Delta_{\nu[\mu\lambda]})$ for $\omega_{\lambda[ab]} = \hat{\omega}_{\lambda[ab]} + \Gamma_{\lambda[ab]}$.

When imposing the transposition invariance to supergravity, one can see from the last two paragraphs that $\Gamma^{\lambda}{}_{[\mu\nu]}$ plays the role of $K^{\lambda}{}_{[\mu\nu]}$ and $\Delta^{\lambda}{}_{[\mu\nu]}$ plays the role of $T^{\lambda}{}_{[\mu\nu]}$ in the field equations.

Appendix B

In this appendix, we reply to common questions about this paper. Unless otherwise specified, we consider that $f(R) = R$, $g_{\mu\nu} = g_{\nu\mu}$ and $\Delta^\lambda{}_{\mu\nu} = \Delta^\lambda{}_{[\mu\nu]}$. *We use the notation of appendix A and also the notation of [1] to denote the "+−" covariant derivatives (see (5) page 579).*

Why consider the old idea of transposition invariance ?
There are many "old ideas" used in modern physics, such as the Kaluza-Klein idea of increasing the number of space dimensions. Imposing the transposition invariance to non-symmetric real expressions is equivalent to imposing the hermiticity to complex expressions, which is widely used in modern physics.

Why denote the torsion-free expressions of General Relativity and supergravity with a hat on top ?
The hat is a symbol of respect used to emphasize the importance of these expressions. The Christoffel symbols are $\hat\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$, the Ricci curvature tensor is $\hat R_{\mu\nu} = \partial_\sigma \hat\Gamma^\sigma_{\mu\nu} - \partial_\mu \hat\Gamma^\sigma_{\sigma\nu} - \hat\Gamma^\sigma_{\mu\tau}\hat\Gamma^\tau_{\sigma\nu} + \hat\Gamma^\tau_{\mu\nu}\hat\Gamma^\sigma_{\sigma\tau}$ and the torsion-free spin connection is $\hat\omega_{\lambda[ab]} = \frac{1}{2}e_a{}^\sigma(\partial_\lambda e_{b\sigma} - \partial_\sigma e_{b\lambda}) - \frac{1}{2}e_b{}^\sigma(\partial_\lambda e_{a\sigma} - \partial_\sigma e_{a\lambda}) - \frac{1}{2}e_a{}^\sigma e_b{}^\tau(\partial_\sigma e_\tau{}^c - \partial_\tau e_\sigma{}^c)e_{c\lambda}$.

Why reject the field equation $\nabla_\lambda g_{\underset{\mu}{-}\underset{\nu}{-}} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma} = 0$ whose solution is $\Gamma^\lambda{}_{\mu\nu} = \hat\Gamma^\lambda{}_{\mu\nu} + K_\mu{}^\lambda{}_\nu$ with $\hat\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$ and $K_{\lambda\mu\nu} = K_{\lambda[\mu\nu]} = \frac{1}{2}(T_{\lambda[\mu\nu]} + T_{\mu[\lambda\nu]} + T_{\nu[\mu\lambda]})$?
First, this equation does not imply $\Gamma^\lambda{}_{(\mu\nu)} = \hat\Gamma^\lambda{}_{\mu\nu}$ when $\Gamma^\lambda{}_{[\mu\nu]} \neq 0$, which complicates the expressions. Secondly, $K^\lambda{}_{[\mu\nu]}$ and $T^\lambda{}_{[\mu\nu]}$ appear indirectly as part of the solution to the equation. Thirdly, why privilege $\nabla_\lambda g_{\underset{\mu}{-}\underset{\nu}{-}}$ over $\nabla_\lambda g_{\underset{\mu}{-}\underset{\nu}{+}}$, $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{-}}$ or $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{+}}$?

Why prefer the field equation $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{-}} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma} = -\Delta_{\lambda[\mu\nu]}$ whose solution is $\Gamma^\lambda{}_{\mu\nu} = \hat\Gamma^\lambda{}_{\mu\nu} + \Gamma^\lambda{}_{[\mu\nu]}$ with $\hat\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$ and $\Gamma_{\lambda[\mu\nu]} = \frac{1}{2}(\Delta_{\lambda[\mu\nu]} + \Delta_{\mu[\lambda\nu]} + \Delta_{\nu[\mu\lambda]})$?
First, this equation implies $\Gamma^\lambda{}_{(\mu\nu)} = \hat\Gamma^\lambda{}_{\mu\nu}$ even when $\Gamma^\lambda{}_{[\mu\nu]} \neq 0$, which simplifies the expressions. Secondly, $K^\lambda{}_{[\mu\nu]}$ is avoided and $\Delta^\lambda{}_{[\mu\nu]}$ appears directly in the equation. Thirdly, the requirement of transposition invariance justifies the use of $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{-}}$ over $\nabla_\lambda g_{\underset{\mu}{-}\underset{\nu}{-}}$, $\nabla_\lambda g_{\underset{\mu}{-}\underset{\nu}{+}}$ or $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{+}}$.

Why use the complicated expression $R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \frac{1}{2}\partial_\mu \Gamma^\sigma_{\sigma\nu} - \frac{1}{2}\partial_\nu \Gamma^\sigma_{\mu\sigma} - \Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\sigma\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\sigma_{(\sigma\tau)}$?
This is to obtain the expression $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{-}} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma}$ in all cases. When $g_{\mu\nu} = g_{\nu\mu}$ and $\Delta^\lambda{}_{\mu\nu} = \Delta^\lambda{}_{[\mu\nu]}$, this "complicated expression" simplifies to $R_{\mu\nu} = \hat R_{\mu\nu} - \Gamma^\tau_{[\mu\sigma]}\Gamma^\sigma_{[\tau\nu]}$.

Why put $\omega_{\lambda[ab]} = \hat\omega_{\lambda[ab]} + \Gamma_{\lambda[ab]}$ in the definition of the Lorentz derivative $D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_c e_\nu{}^c$?
This is to obtain the relation $D_\mu e_{\nu a} - D_\nu e_{\mu a} = \Gamma_{\mu[a\nu]} - \Gamma_{\nu[a\mu]} = \Delta_{a[\mu\nu]} \Rightarrow \Gamma_{\mu[\lambda\nu]} + \Gamma_{\nu[\mu\lambda]} = \Delta_{\lambda[\mu\nu]}$. In other words, $\omega_{\lambda[ab]} = \hat\omega_{\lambda[ab]} + \Gamma_{\lambda[ab]}$ is merely an ansatz used to obtain the antisymmetric part of the field equation $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{-}} = -\Delta_{\lambda[\mu\nu]}$.

Why leave undefined the covariant derivative ∇_μ of the vielbein $e_\nu{}^a$?
The torsion-free covariant derivative $\hat\nabla_\mu e_\nu{}^a = \hat D_\mu e_\nu{}^a - \hat\Gamma^\sigma{}_{\mu\nu} e_\sigma{}^a = \partial_\mu e_\nu{}^a + \hat\omega_\mu{}^a{}_c e_\nu{}^c - \hat\Gamma^\sigma{}_{\mu\nu} e_\sigma{}^a = 0$ and the Lorentz derivative $D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_c e_\nu{}^c$ with $\omega_{\lambda[ab]} = \hat\omega_{\lambda[ab]} + \Gamma_{\lambda[ab]}$ are sufficient to obtain the field equation $\nabla_\lambda g_{\underset{\mu}{+}\underset{\nu}{-}} = -\Delta_{\lambda[\mu\nu]}$. Nevertheless, following [1](5) page 579, it is always possible to define the "+−" covariant derivatives $\nabla_\mu e_{\underset{\nu}{+}}{}^a = D_\mu e_\nu{}^a - \Gamma^\sigma{}_{\nu\mu} e_\sigma{}^a$ and $\nabla_\mu e_{\underset{\nu}{-}}{}^a = D_\mu e_\nu{}^a - \Gamma^\sigma{}_{\mu\nu} e_\sigma{}^a$.

References

[1] A. Einstein, "A Generalization of the Relativistic Theory of Gravitation", Annals of Mathematics, Vol. 46, No. 4, p578-584, 1945.

[2] A. Einstein and E. G. Straus, "A Generalization of the Relativistic Theory of Gravitation, II", Annals of Mathematics, Vol. 47, No. 4, p731-741, 1946.

[3] A. Einstein and B. Kaufman, "A New Form of the General Relativistic Field Equations", Annals of Mathematics, Vol. 62, No. 1, p128-138, 1955.

[4] A. Einstein, "The meaning of Relativity", Fifth Edition : Including the Relativistic Theory of the Non-Symmetric Field, Princeton University Press, p109-166, 2004. ISBN : 0-691-12027-7.

[5] J. W. Moffat, "Nonsymmetric Gravitational Theory", 1994.
See http://arxiv.org/ gr-gc/9411006. See also gr-gc/9411027 and gr-gc/9412009.

[6] T. P. Sotiriou and V. Faraoni, "$f(R)$ theories of gravity", 2008.
See http://arxiv.org/ gr-gc/0805.1726. See also gr-gc/0604006 and gr-gc/0611040.

[7] M. Trodden and S. M. Carroll, "TASI Lectures : Introduction to Cosmology", 2008.
See http://arxiv.org/ astro-ph/0401547v1.

[8] P. van Nieuwenhuizen, A. Rebhan, D. V. Vassilevich and R. Wimmer, "Boundary Terms in Supergravity and Supersymmetry", 2008.
See http://arxiv.org/ hep-th/0606075v2.